

Question 2: How are derivatives used to compute elasticity?

In economics, the term elasticity refers to the responsiveness of one economic variable to changes in another economic variable. The elasticity is measured in terms of percentage changes instead of absolute changes. This means we measure the change in a variable as a percentage of the original amount of the variable. For instance, the percent change in a variable X is defined as

$$\text{Percent change in } X = \frac{\text{Change in the variable } X}{\text{Original value of } X}$$

This definition can be symbolized in a compact form by symbolizing the change as ΔX .

If a variable X changes from one value X to another value $X + \Delta X$, then

$$\text{Percent change in } X = \frac{\Delta X}{X}$$

Suppose that the value of X changes from 20 to 30. The percentage change is

$$\text{Percent change in } X = \frac{30 - 20}{20} = 0.5$$

Written as a percentage, this is a percentage change in X of 50%.

If we wish to find the elasticity of Y with respect to X , we find the ratio of the percentage change in Y to the percentage change in X .

The elasticity of Y with respect to X is

$$\text{Elasticity of } Y \text{ with respect to } X = \frac{\text{Percent change in } Y}{\text{Percent change in } X}$$

For most elasticity calculations, the percent change in one variable corresponds to an increase while the other percent change corresponds to a decrease. This means the numerator and denominator will have opposite signs resulting in a negative value. Some textbooks will define the elasticity with a negative sign or in absolute values, but we'll preserve the sign to emphasize the nature of the percent changes. An elasticity more negative than -1 indicates that a percent increase in X corresponds to a greater percent decrease in Y . In this case, we would say that the variable Y is elastic with respect to variable X .

An elasticity between -1 and 0 indicates that a percent increase in X corresponds to a smaller percent decrease in Y . In this case, the variable Y is inelastic with respect to the variable X .

An elasticity of -1 means that a percent increase in X corresponds to an identical percent decrease in the variable Y . In this case, the variable Y is said to be unit-elastic with respect to the variable X .

The most common use of elasticity in economics is price elasticity of demand or elasticity of demand with respect to price. This concept allows us to explore the responsiveness of the consumer demand for some product or service to changes in the price of that product or service. If the price of a cup of coffee were to increase, how would this effect the quantity sold? In 2006, Starbucks increased prices on all coffee drinks by 5 cents. At one franchise in Sacramento, California, the price on a tall coffee increased from \$1.65 to \$1.70. As expected, the demand for tall coffees dropped from 440 units per day to 436 units per day. The percent change in price P is

$$\frac{\Delta P}{P} = \frac{1.70 - 1.65}{1.65} \approx .03$$

or about 3%. The percent change in the demand Q is

$$\frac{\Delta Q}{Q} = \frac{436 - 440}{440} \approx -.009$$

or about -0.9%. An 3% increase in price corresponds to a 0.9% decrease in the quantity of tall coffees sold.

The elasticity of demand with respect to price or price elasticity of demand is

$$E = \frac{\text{Percent change in } Q}{\text{Percent change in } P} = \frac{-0.009}{.03} = -0.3$$

Since this is a ratio, we can think of it as $\frac{-0.3}{1}$ and say that a 1% increase in price corresponds to a 0.3% drop in demand for tall coffee. On the other hand, we could also think of this ratio as $\frac{0.3}{-1}$ and say that a 1% drop in price corresponds to a 0.3% increase in demand. Many economics textbooks work in terms of larger price changes and will interpret this ratio as $\frac{-3}{10}$. In this case, we would say that a 10% increase in the price of a tall coffee is accompanied by a 3% decrease in demand.

If the relationship between demand and price is given by a function, we can utilize the derivative of the demand function to calculate the price elasticity of demand. If we write think of the change in price as being from P to $P + \Delta P$ and the corresponding change in demand as being from Q to $Q + \Delta Q$, the corresponding percent changes in price and demand are $\frac{\Delta P}{P}$ and $\frac{\Delta Q}{Q}$. The definition of elasticity leads to

$$E = \frac{\text{Percent change in } Q}{\text{Percent change in } P} \quad \text{Definition of price elasticity of demand}$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} \quad \text{Replace the percent changes with appropriate symbols}$$

$$= \frac{\Delta Q}{Q} \frac{P}{\Delta P} \quad \text{Dividing by } \frac{\Delta P}{P} \text{ is the same as multiplying by the reciprocal } \frac{P}{\Delta P}$$

$$= \frac{P}{Q} \frac{\Delta Q}{\Delta P} \quad \text{Rearrange the factors in the numerators of each fraction.}$$

The factor $\frac{\Delta Q}{\Delta P}$ represents the average rate of change of demand Q with respect to price P . If we assume the change in price is small, we can replace the average rate of change with the instantaneous rate of change, $\frac{dQ}{dP}$.

If the demand Q and the price P are related by a demand function $Q = f(P)$, then the price elasticity of demand E is

$$E \approx \frac{P}{Q} \frac{dQ}{dP}$$

If $E < -1$, then demand is elastic and a percent increase in price yields a larger percent decrease in demand.

If $-1 < E < 0$, then demand is inelastic and a percent increase in price yields a smaller percent decrease in demand.

Several other cases of elasticity should be mentioned. If $E = -1$, then the demand is unit-elastic and a percent increase in price yields the same percent decrease in demand. Demand is perfectly elastic if any increase in price causes the quantity demanded to decrease to 0. Demand is perfectly inelastic if and increase or decrease in price makes no change in the quantity demanded.

Some economists prefer to define elasticity as a positive number by taking the absolute value of $\left| \frac{P}{Q} \frac{dP}{dQ} \right|$. This change loses the idea that an increase in price leads to a drop in the quantity sold, For this reason, we'll use the definition above for computing price elasticity of demand.

Example 3 Calculate the Price Elasticity of Demand

Using data from market years 2000 through 2010, the relationship between the price per bushel of oats and the number of bushels of oats sold is given by

$$Q = 152.07 P^{-0.543}$$

where P is in dollars and Q is in millions of bushels.

- a. Find the price elasticity of demand when the price per bushel is \$1.75.

Solution Since the relationship between the quantity and price is given as a function of price, we may compute the price elasticity of demand from

$$E = \frac{P}{Q} \frac{dQ}{dP}$$

The derivative $\frac{dQ}{dP}$ is found from the relationship between quantity and price. Using the Product with a Constant Rule and the Power Rule, we get

$$\begin{aligned} \frac{dQ}{dP} &= \frac{d}{dP} [152.07 P^{-0.543}] \\ &= 152.07 (-0.543 P^{-1.543}) \\ &= -82.574 P^{-1.543} \end{aligned}$$

Substitute the expression $\frac{dQ}{dP} = -82.574 P^{-1.543}$ for the derivative and the quantity $Q = 152.07 P^{-0.543}$ into the elasticity formula yields

$$\begin{aligned}
 E &= \frac{P}{Q} \frac{dQ}{dP} \\
 &= \frac{P}{152.07P^{-0.543}} (-82.574P^{-1.543}) \\
 &\approx -0.543
 \end{aligned}$$

Since the elasticity simplifies to a form that does not contain P , the elasticity does not depend upon price.

- b. Is the demand for oats elastic or inelastic when the price per bushel is \$1.75?

Solution Since the elasticity, $E \approx -0.543$, is between -1 and 0, the elasticity is inelastic. As a consequence, a change in price of 1% will result in a 0.543% decrease in the quantity demanded.



The product of the price per unit and the quantity demanded is equal to the revenue from the product. If the quantity demanded decreases as the price is increased, what will be the overall effect on the total revenue? To answer this question, let's examine a table of prices, corresponding quantities, revenue and the elasticity for a product where the demand function is $Q = -P + 120$.

<i>price P</i>	20	40	60	80	100
<i>quantity Q</i>	100	80	60	40	20
<i>total revenue TR</i>	2000	3200	3600	3200	2000
<i>price elasticity of demand E</i>	-0.2	-0.5	-1	-2	-5

Even though the slope of the demand curve is -1, the price elasticity of demand changes as the price increases. For lower prices (green), the price elasticity of demand is between -1 and 0 and increasing the price increases revenue. For higher prices

(blue), the elasticity is more negative than 1 and increasing the price leads to lower revenue. The highest revenue occurs when the elasticity is equal to -1.

Should prices be increased?

1. If demand is inelastic ($-1 < E < 0$), then a price increase yields an increase in total revenue.
2. If demand is elastic ($E < -1$), then a price increase yields a decrease in total revenue.
3. Total revenue is highest at a price where demand is unit-elastic.

For inelastic demand, price increases are countered by small decreases in the quantity resulting in more revenue. However, when demand is elastic, price increases lead to large drops in the quantity sold that lowers the overall revenue. If maximizing revenue is the overall goal, setting the price where the price elasticity of demand is equal to -1 is ideal. This goal ignores cost and should be approached cautiously.

Example 4 Calculate the Price Elasticity of Demand

A small technology company in Northern California is developing a tablet PC to compete with Apple's Ipad. Based on market surveys, the company believes the quantity Q (in thousands of units) that will be demanded by consumers is related to the price P (in thousands of dollars) by the relationship

$$Q = 4000 - 250P^2$$

- a. Find the price elasticity of demand when the price of the tablet PC is \$3000.

Solution To compute the price elasticity of demand, we need to find the derivative of the demand function $Q = 4000 - 250P^2$:

$$\begin{aligned}\frac{dQ}{dP} &= \frac{d}{dP}[4000 - 250P^2] \\ &= \frac{d}{dP}[4000] - 250 \frac{d}{dP}[P^2] && \text{Use the Sum / Difference Rule and the Constant Times a Function Rule} \\ &= 0 - 250(2P) && \text{Use the Power Rule} \\ &= -500P\end{aligned}$$

Use the derivative and the expression for the quantity Q in the formula for price elasticity of demand,

$$\begin{aligned}E &\approx \frac{P}{Q} \frac{dQ}{dP} \\ &\approx \frac{P}{4000 - 250P^2} (-500P)\end{aligned}$$

We can simplify the right hand side of this equation to make it easier to substitute the price in:

$$\begin{aligned}E &\approx \frac{P(-500P)}{4000 - 250P^2} && \text{Combine factors} \\ &\approx -\frac{500P^2}{4000 - 250P^2} && \text{Simplify numerator}\end{aligned}$$

A tablet PC priced at 3000 dollars corresponds to $P = 3$. If we put this value into the expression for price elasticity of demand, we get

$$E \approx -\frac{500(3)^2}{4000 - 250(3)^2} \approx -2.57$$

Since the elasticity, $E \approx -2.57$, is more negative than -1, the demand is elastic. As a consequence, revenue will drop as the price increases.

b. At what price is the price elasticity of demand equal to -1?

Solution From part a, we know that a price of \$3000 is elastic. At lower prices the demand might be inelastic. To find the dividing line between elastic and inelastic, we set the price elasticity of demand equal to -1,

$$-\frac{500P^2}{4000 - 250P^2} = -1$$

Now solve for P :

$$(4000 - 250P^2) \frac{-500P^2}{4000 - 250P^2} = -1(4000 - 250P^2)$$

Multiply both sides by the denominator to clear fraction

$$-500P^2 = -4000 + 250P^2$$

Add $250P^2$ to both sides

$$-750P^2 = -4000$$

Divide both sides by 750 and reduce

$$P^2 = \frac{16}{3}$$

$$P = \pm \sqrt{\frac{16}{3}}$$

Square root both sides of the equation

The negative price is not a possible value for the price of a tablet PC.

The other price, $P = \sqrt{\frac{16}{3}}$, is approximately 2.309. Since P has units of thousands of dollars, this value corresponds to a price of \$2309.

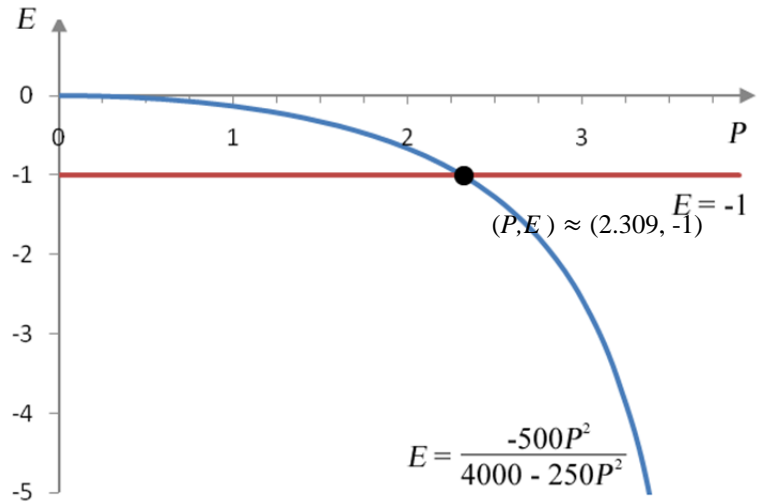


Figure 4 – At the point of intersection, the elasticity is equal to -1. To the left or right of the point of intersection, the price elasticity of demand is either lower or higher than -1.

If the price is lower than \$2309, the value of E is lower than -1 meaning demand is inelastic. Prices higher than \$2309 result in elastic demand. This means revenue is maximized at a price of \$2309.